

Virial estimator for dark energy

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A new estimator of the local density of dark energy is suggested which comes from the virial theorem for non-relativistic gravitating systems embedded in the uniform dark energy background.

1 Introduction

The virial theorem of Newtonian mechanics plays a central part in the studies of gravitationally bound quasi-stationary astronomical many-body systems like star clusters, galaxies, their groups and clusters. The theorem is widely used for the mass determination of the systems. Most impressive examples of this kind are the discovery of dark matter in the Coma cluster by Zwicky [1] and the discovery of dark haloes of galaxies by Einasto et al.[2]. In this note we show that the virial theorem can also be used for detection and measurement of dark energy, or Einstein's cosmological constant Λ .

2 Dark energy in terms of Newtonian mechanics

The current standard cosmological Λ CDM model treats dark energy macroscopically as a vacuum-like medium with density $\rho_{\Lambda, glob} = 8\pi G\Lambda > 0$ (where G is the Newton gravitational constant). The density is perfectly uniform and its value is the same in any reference frame. Therefore ρ_{Λ} is the only physical parameter of the medium. In the Λ CDM cosmology, the antigravity produced by dark energy controls the global cosmological expansion on the scales which are larger than the size of the cosmic cell of uniformity ($\simeq 300 - 1000$ Mpc). Though the isotropic Λ CDM model does not extend to local volumes which are located deeply inside the cosmic cell of uniformity, the interpretation of dark energy adopted in the model may reasonably be extrapolated as well to studies of local effects of dark energy. Under this assumption, we have demonstrated earlier that dark energy dominates the dynamics of local flows of expansion observed around nearby groups of galaxies and the nearest Virgo cluster [3-12] on the scales 1-20 Mpc. It was also found that the local dark energy density ρ_{Λ} is nearly or even exactly coincides with the global value $\rho_{\Lambda, glob}$. Following this approach here, we will nevertheless discriminate, generally, between the local ρ_{Λ} and global $\rho_{\Lambda, glob}$ values of dark energy.

Dark matter and cosmic baryonic matter are non-relativistic and can be described with the use of the classic Newtonian mechanics. As for dark energy, it is an essentially relativistic substance, whatever its physical nature and microscopical structure might be; it can adequately be described only in terms of general relativity. It is well-known nevertheless that dark energy may be put in the context of Newtonian mechanics (as it is explained, for instance, in [3] and Sec. 3.4 of [4]; see also [13,14]). As a medium dark energy is characterized by positive density, ρ_{Λ} and pressure p_{Λ} which is negative. They are related to each other by the equation of state, $\rho_{\Lambda} = -p_{\Lambda}$ (hereafter the speed of light $c = 1$). The medium with that equation of state is vacuum, in terms of mechanics since rest and motion are not discriminated relative to this medium.

According to general relativity, the source of gravity is not matter density ρ alone (as it is in the Newtonian mechanics), but energy-momentum tensor which includes both density ρ and pressure p . As a result, an "effective gravitating density" ρ_{eff} is introduced which is simply the sum $\rho + 3p$, if the medium is uniform. This may directly be seen in the Friedmann ("first") equation, as well as in the de Sitter solution and some other general relativity relations. For dark energy with its equation of state $\rho_\Lambda = -p_\Lambda$ one has:

$$\rho_{eff} = \rho_\Lambda + 3p_\Lambda = -2\rho_\Lambda < 0. \quad (1)$$

Negative effective density means that dark energy produces a repulsion force, or antigravity.

This result is borrowed from general relativity to Newtonian mechanics (see in Secs.2.1 and 3.4 of [4]). Consequently, in Newtonian description, dark energy antigravity is represented by the repulsive antigravity force (per unit mass)

$$\mathbf{F}_\Lambda = +\frac{8\pi}{3}G\rho_\Lambda\mathbf{r}, \quad (2)$$

where the radius-vector \mathbf{r} denotes the spatial position of a particle. The corresponding potential

$$\Phi_\Lambda = -\frac{4\pi}{3}G\rho_\Lambda r^2. \quad (3)$$

The relations (2),(3) come directly from the Schwarzschild-de Sitter spacetime considered in the limit of the weak deviations from the Galilean metric (see again section 3.4 of [4]). This approximation is quite appropriate to the physical conditions in groups and clusters of galaxies on the spatial scales 1-20 Mpc we have here in mind, since $|\Phi_\Lambda|/c^2 \sim (0.3 - 1) \times 10^{-6} \ll 1$. Because of the same reason, the local curvature effects are considered negligible. Note also that the general cosmological expansion does not affect the dynamics of local quasi-stationary systems like groups and clusters; because of this the Friedmann cosmological equations are not involved in the description of local dark energy effects.

General relativity indicates also that the inertial and passive gravitational mass of dark energy (the both are $\rho_\Lambda + p_\Lambda$, per unit mass) are zero; it means in particular that dark energy is not affected by its own antigravity and therefore its self-gravity potential energy is zero (which is important for considerations of the next section).

3 Modified virial theorem

The first versions of the virial theorem with Λ were proposed decades ago [15,16]; more recently various aspects of the modified theorem have been discussed in [12,17-20]. Here we show briefly how the theorem may be deduced with the use of Eqs.(2), (3).

As is long known, a finite collection of point-mass particles interacting gravitationally via the Newtonian potential $\Phi_M \propto 1/r$ and bounded in space obeys the virial relation $\langle K \rangle = -\frac{1}{2} \langle U \rangle$, where $\langle K \rangle$ is the time average of the total kinetic energy of the system, and $\langle U \rangle$ is the time average of its total potential energy. Generally, the virial theorem states that if the potential is a homogeneous function of order n of the coordinates, then the average kinetic energy $\langle K \rangle$ and average potential energy $\langle U \rangle$ of the system relate to each other as $\langle K \rangle = \frac{n}{2} \langle U \rangle$.

If a system of N gravitating particles is embedded in the uniform dark energy background, the particles move in the potential which is the sum of two components: the Newtonian one Φ_M with $n = -1$ for the particle-particle interaction and the dark energy potential Φ_Λ with $n = 2$ (see Eq.(3)) for the particle-dark energy interaction. Therefore one may expect that a modified virial theorem for the sum of the potentials must include two items: the conventional $-\frac{1}{2} \langle U \rangle$ and a new one $+ \langle W \rangle$, where W is the potential energy of the particles which is due to their interaction with the dark energy background (see below).

Let us remind (see references in Sec.2) that the generalized virial relation that accounts for both matter gravity and dark energy antigravity can directly be

obtained from the Newtonian mechanics of an arbitrary bound N -body system with masses m_i , where $i = 1, 2, 3, \dots, N$. The potential energy of the interaction between the i -th and the j -particles

$$U_{ij} = -\frac{Gm_i m_j}{r_{ij}}, \quad (4)$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The particle-particle contribution to the total potential energy of the system is the sum over all N particles:

$$U = -\frac{1}{2} \sum_i^N \sum_{j \neq i}^N \frac{Gm_i m_j}{r_{ij}}. \quad (5)$$

The potential energy of the interaction of the i -th particle with the dark energy background (see Eq.(3))

$$W_i = m_i \Phi_\Lambda = -m_i \frac{4\pi}{3} G \rho_\Lambda r^2. \quad (6)$$

The contribution of the particle-dark energy interaction to the total potential energy

$$W = -\sum_i^N m_i \frac{4\pi}{3} G \rho_\Lambda r^2. \quad (7)$$

According to Eq.(2), the equation of motion for the i -th particle moving with a non-relativistic velocity

$$m_i \ddot{\mathbf{r}}_i = -\sum_{j \neq i}^N \frac{Gm_i m_j}{r_{ij}^3} (\mathbf{r}_i - \mathbf{r}_j) + 2m_i \frac{4\pi}{3} G \rho_\Lambda \mathbf{r}_i. \quad (8)$$

The scalar product of both sides of this equation by \mathbf{r}

$$m_i (\mathbf{r}_i \ddot{\mathbf{r}}_i) = -\sum_{j \neq i}^N \frac{Gm_i m_j}{r_{ij}^3} (\mathbf{r}_i - \mathbf{r}_j) \mathbf{r}_i + 2m_i \frac{4\pi}{3} G \rho_\Lambda \mathbf{r}_i^2. \quad (9)$$

Having in mind that the particle velocity $\mathbf{v}_i = \dot{\mathbf{r}}_i$ and also

$$\frac{d}{dt}(\mathbf{v}_i \mathbf{r}_i) = \mathbf{v}_i^2 + \dot{\mathbf{v}}_i \mathbf{r}_i, \quad (10)$$

we rewrite the left side of Eq.(9):

$$m_i \frac{d}{dt}(\mathbf{v}_i \mathbf{r}_i) - 2K_i, \quad (11)$$

where $K_i = \frac{1}{2}m_i v^2$ is the kinetic energy of the particle. The sum over i and a simple identical transformation of Eq.(9) (about the transformation of the first item in the right side see, for example, [21]) lead to

$$\frac{1}{2}m_i \frac{d}{dt}(\mathbf{v}_i \mathbf{r}_i) - K = \frac{1}{2}U - W, \quad (12)$$

The potential energies U and W are given by Eqs.(5),(7). The average over a large time of the first item in the left side here is zero, if the particles are located in a bound finite volume. Then we finally find:

$$K = -\frac{1}{2}U + W. \quad (13)$$

This is the modified virial theorem for a N-body quasi-stationary bound system embedded in the dark energy background. The result shows how the classic relation which treats each of the potential energies separately (see above) reveals itself in real astronomical systems: the virial relation contains two items in the right side, the first of them (positive) is conventional, and the second one (negative) is due to the dynamical effects of dark energy.

4 Theorem for one particle

The generalized virial relation of Eq.(13) may be derived for one light particle of mass m moving in the central field of a heavy mass M along a finite bound orbit in the dark energy background. The mass M produces the potential $\Phi_M = -GM/r$ and the dark energy background produces (as above) the potential $\Phi_\Lambda = -\frac{4\pi}{3}G\rho_\Lambda r^2$. The particle equation of motion

$$\dot{\mathbf{v}} = -\frac{GM}{r^2} \frac{\mathbf{r}}{r} + \frac{8\pi}{3}G\rho_\Lambda \mathbf{r}. \quad (14)$$

Multiplying both sides of Eq.(14) by \mathbf{r} , one gets:

$$\dot{\mathbf{v}} \mathbf{r} = -\frac{GM}{r} + \frac{8\pi}{3}G\rho_\Lambda r^2 = \Phi_M - 2\Phi_\Lambda. \quad (15)$$

The transformation of Eq.(10) applied to Eq.(15) leads now to

$$\frac{d}{dt}(\mathbf{vr}) = v^2 - \frac{GM}{r} + \frac{8\pi}{3}G\rho_\Lambda r^2 = v^2 + \Phi_M - 2\Phi_\Lambda. \quad (16)$$

Or in terms of energies $K = \frac{1}{2}mv^2, U = m\Phi_M, W = m\Phi_\Lambda$:

$$\frac{1}{2}\frac{d}{dt}(\mathbf{vr}) = K + \frac{1}{2}U - W. \quad (17)$$

Averaging over time gives again the virial relation in form of Eq.(13) which is valid now for the finite motion of one particle as well as for the bound N-body system.

The particle potential energy W due to dark energy is an essentially negative value. Because of this, the kinetic energy of a virialized system is less in the presence of the dark energy background than in the perfectly empty space. The physical cause is clear: the dark energy antigravity cancels partly the matter gravity; as a result, the potential well of the system is not as deep as it would be in empty space. Accordingly, the velocities of the particles are less in this case. Since the kinetic energy is positive, Eq.(13) implies that

$$|U| > 2|W|. \quad (18)$$

This inequality is the necessary condition for the existence of a system with finite bound orbits. Eq.(18) guaranties that gravity is stronger than antigravity in the volume of the system. Indeed, if a steady-state characteristic radius of a system is r , and its total mass is M , then $|U| = GM^2/r$ and $2|W| = M\frac{8\pi}{3}G\rho_\Lambda r^2$. Then Eqs.(13),(18) give:

$$r < R_\Lambda = [\frac{3M}{8\pi\rho_\Lambda}]^{1/3} \simeq 1 \times (M_{12})^{1/3} Mpc. \quad (19)$$

Here R_Λ is the "zero-gravity radius"[6,7]: at the distance $r = R_\Lambda$ the gravity and antigravity of the system completely balance each other. In Eq.(19), M_{12} is the mass in the units of $10^{12}M_\odot$, and the observed value $\rho_\Lambda = 7.5 \times 10^{-30} \text{ g cm}^{-3}$ is used.

Inequality (19) has a rather universal sense: the size of any gravitationally bound system must obey it in the world of dark energy.

5 Virial estimators

The relation of Eq.(13) provides a mass estimator which generalizes the conventional virial estimator:

$$M = \frac{v^2 r}{G} + \frac{8\pi}{3} \rho_\Lambda r^3, \quad (20)$$

where r and v are characteristic size and velocity of the system. Eq.(20) shows that the mass of astronomical system is systematically underestimated, if dark energy is not taken into account.

The effect of dark energy is given by the new positive item in the right side of Eq.(20). The term is quantitatively equivalent to the effective mass of dark energy contained in the volume of the system. The physical sense of the extra term is clear. Indeed, the antigravity of dark energy cancels partly the gravity of dark matter and baryons in the volume of the system. Consequently, the characteristic velocity v of the first term in the right side of Eq.20 represents the partly cancelled gravity of dark matter and baryons, not their total gravity. The second term in the right side of Eq.(20) "restores" the true mass of the dark matter and baryons. The mass correction is given by the relation $M_{vir} = \frac{v^2 r}{G}(1 + f)$, where f is the ratio of the new item to the conventional one:

$$f \equiv \frac{8\pi}{3} G \rho_\Lambda (r/v)^2 = \left(\frac{t_{cross}}{t_\Lambda}\right)^2. \quad (21)$$

Here $t_{cross} = r/v$ is the system crossing time; $t_\Lambda = (\frac{8\pi G}{3} \rho_\Lambda)^{-1/2} = 16$ Gyr is the dark energy characteristic time which is near the current age of the Universe. The ratio f is less than unity for any gravitationally bound system:

$$f = \left(\frac{t_{cross}}{t_\Lambda}\right)^2 \simeq 0.4 \left(\frac{r}{1Mpc}\right)^2 \left(\frac{v}{100km/s}\right)^{-2}. \quad (22)$$

Having in mind all the necessary reservations regarding a degree of virialization of a system, its age, shape, internal structure, the way in which available data are obtained, etc., we may illustrate the dark energy effect in Eq.(20) by simplest examples. For a halo of giant galaxy like the Milky Way or M31 with $r \simeq 0.3$ Mpc, $v \simeq 200$ km s⁻¹, so that the ratio $f \simeq 0.03$; for a typical sparse group of galaxies like the Local Group with $r \simeq 1$ Mpc, $v \simeq 70$ km s⁻¹, the ratio $f \simeq 0.8$, and for a big cluster of galaxies with $r \simeq 5$ Mpc, $v \simeq 1000$ km s⁻¹, the ratio $f \simeq 0.04$. These figures show that the effect of dark energy is most prominent for groups of galaxies: the correctly estimated mass is 30-50% larger than that made with the traditional virial estimator.

Eq.(20) shows that

$$\rho_\Lambda = \frac{3}{8\pi G} \left[\frac{GM}{r^3} - \frac{v^2}{r^2} \right]. \quad (23)$$

If the matter mass M of a system, its characteristic size r and velocity v are independently measured, the system may serve as a measurement setup for local (on the spatial scale of the system) dark energy detection, and Eq.(23) turns out to be an estimator of the density ρ_Λ . In terms of the crossing time t_{cross} , cosmological critical density $\rho_c = \frac{3}{8\pi G} H^2$ and the Hubble factor H the estimator takes the form:

$$\rho_\Lambda = \frac{1}{2} \langle \rho \rangle - \rho_c \frac{1}{(Ht_{cross})^2}, \quad (24)$$

where $\langle \rho \rangle = \frac{M}{\frac{4\pi}{3}r^3}$ is the mean density of the system. Or in the units of the critical density:

$$\Omega_\Lambda = \frac{1}{2} \langle \Omega \rangle - (Ht_{cross})^{-2}, \quad (25)$$

where $\langle \Omega \rangle = \frac{\langle \rho \rangle}{\rho_c}$.

For a galaxy group with the mass $M = (1-2) \times 10^{12} M_\odot$, the size $r = 1$ Mpc and the velocity $v = 70$ km/s, we have $\langle \Omega \rangle = 3-6$ and the product $Ht_{cross} \simeq 1$. Then the estimator of Eq.(25) gives:

$$\Omega_\Lambda = 0.5 - 2. \quad (26)$$

The interval of possible values is not too wide in Eq.(26), but it contains comfortably the value $\Omega_{\Lambda glob} = 0.70 - 0.75$ which is known from the global cosmological observations [22]. The result implies that the local density of dark energy is near the global value, if not coincides with it exactly, – in complete agreement with our earlier findings (see Sec.2).

6 Conclusions

To summarize, the modified virial theorem provides us with a modified mass estimator and – which is perhaps more interesting – a new estimator for the dark energy density on the local scale of groups and clusters of galaxies. The virial mass determination has proved its usefulness in decades of studies. However practical effectiveness of the virial estimator for dark energy is not yet so obvious; the theoretical recipe we suggest for virial detection and measurement of local dark energy needs more empirical studies.

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